Introduction to descent theory

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Disclaimers

- All definitions are quasi-definitions.
- All constructions are quasi-constructions.

This means they are correct but not quite.

Introduction

A prototypical situation in geometry

Here's an imaginary situation in geometry (but which arises often enough in algebraic geometry!).

Let X be a topological space. Let $\bigcup_{i \in I} U_i$ be a covering. Assume that for each U_i we are given some geometric data in the form of a 'functorial' assignment $U_i \mapsto \mathcal{C}(U_i)$ of sets or categories.

Question: Can we understand the geometric data on X from the information contained in the $\mathcal{C}(U_i)$?

Does this data glue together on the U_i to give us data on X? Or in other words the assignment $U \mapsto \mathcal{C}(U_i)$ satisfy descent?

A prototypical situation in geometry

Answer: It depends on what sort of data we are trying to glue.

Let me give a few examples for culture.

1. If X is a topological space and $\mathcal{F}$ is a sheaf on X, then it tautologically satisfies descent.
2. If X is as above, then the assignment $X \mapsto \mathrm{Vec}_n(X)$ i.e. isomorphism classes of vector bundles of rank n will not satisfy descent.
3. In the same situation as above, assignment $X \mapsto \mathbf{Vec}_n(X)$ i.e. the *groupoid* of rank n vector bundles will satisfy descent. It is a *categorification* of the previous example.

A prototypical situation in geometry

- The failure of the functor $\mathrm{Vec}_n(-)$ to satisfy descent is really telling us that isomorphism classes is a truncated invariant.
- It remembers stuff which is isomorphic but doesn't remember why.
- In this talk we will study why exactly this failure occurs in the case of $\mathrm{Vec}_1(-)$ and how studying $\mathbf{Vec}_1(-)$ will correct the situation.
- Before we do that we need to make gluing precise.

Reminder on Sheaves, etc

Sheaves, etc

- Associate to X a category $\mathrm{Op}(X)$, the *topological site* of X.
- Objects of $\mathrm{Op}(X)$ are open sets $U \subset X$ and we think of these objects as coming with the data of the canonical inclusion $j\colon U \hookrightarrow X$, although we will abuse notation by just writing U for $j\colon U \hookrightarrow X$.

Sheaves, etc

If $V \subset U$ are open in X, then an arrow between V and U is given by the diagram of inclusions

$$\begin{tikzcd} V \arrow[rr, "j''"] \arrow[dr, "j'"'] & & U \arrow[dl, "j"] \\ & X & \end{tikzcd}$$

If V is not contained in or equal to U, then $\mathrm{Hom}(V, U) = \emptyset$.

Sheaves, etc

- A *presheaf* on $\mathrm{Op}(X)$ is just a contravariant set valued functor on X.
- If $F\colon \mathrm{Op}(X)^{op} \to \mathrm{Sets}$ is a presheaf and $U \in \mathrm{Op}(X)$, then elements of $F(U)$ are called sections of F over U.
- If F is a presheaf on $\mathrm{Op}(X)$ then for any morphism

$$j\colon V \to U$$

in $\mathrm{Op}(X)$ there is an induced morphism

$$j^*\colon F(U) \to F(V)$$

which we call the *restriction* morphism.
- The image of $s \in F(U)$ under the restriction map $j^*\colon F(U) \to F(V)$ is denoted $s|_V$ to correspond with the psychological notion of restriction.

Sheaves etc

- For $U \in \mathrm{Op}(X)$ a *covering of U* in $\mathrm{Op}(X)$ is a collection of morphisms $\{U_i \to U\}_{i \in I}$ so that the induced morphism $j \colon \coprod_{i \in I} U_i \to U$ is surjective.
- A presheaf F is a *sheaf* if for all $U \in \mathrm{Op}(X)$ and for every covering $\{U_i \to U\}_{i \in I}$ of U, the following *unique gluing* or *descent* condition is satisfied.

  Descent condition/ Unique gluing condition: If $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_j \cap U_i}$ for all $i, j \in I$ then there exists a unique section $s \in F(U)$ so that $s|_{U_i} = s_i$ for all $i \in I$.

  Motto-Sheaves are presheaves in which sections glue uniquely

  Fancy motto-Sheaves are presheaf which satisfy descent.

Very Important example of a sheaf- Let Y be any reasonable topological space. Then the presheaf $\mathrm{Maps}(-, Y)$ is a sheaf on $\mathrm{Op}(X)$.

The reason is that 'continuous maps glue'.

Motto-Representable presheaves are sheaves.

Line bundles and the functor Pic

Vector bundles

- Recall that a *complex vector bundle* of rank n on a (connected) topological space X is the data of a surjective morphism $\pi\colon E \to X$ so that for each point $x \in X$, the fiber E_x is a complex vector space of dimension n.
- We further require *local triviality* i.e. there is an open covering $\bigcup_{i\in I} U_i = X$ so that for each $i \in I$ the restriction $E|_{U_i}$ satisfies $E|_{U_i} \cong U_i \times \mathbf{C}^n$.
- A vector bundle on X of rank n is trivial if it is isomorphic to the projection
$$\pi\colon X \times \mathbf{C}^n \to X.$$
- A *line bundle* is a vector bundle of rank 1.

Vector bundles

- A morphism of vector bundles over X is a commuting diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{id_X} & X \end{array}$$

  which restricts to a linear map on the fiber.

- If $f : X \to Y$ is a continuous morphism and E is a vector bundle of rank n on Y, then the pullback

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

  is a vector bundle of rank n on X and preserves isomorphism classes.

Line bundles

- Thus we can define the functor $\mathrm{Vec}_n(-)\colon \mathrm{Top} \to \mathrm{Sets}$ which sends a topological space to the set of isomorphism classes vector bundles of rank n over it.
- This is contravriantly functorial in it's argument and defines a presheaf on $\mathrm{Op}(X)$.
- We are interested in the case $n = 1$ and write the functor as $\mathrm{Pic}\colon \mathrm{Top} \to \mathrm{Sets}$ which sends

$$X \rightsquigarrow \{\text{Iso classes of line bundles on } X\}$$

Line bundles

Question: We want to ask if $\mathrm{Pic}(-)$ is a sheaf on $\mathrm{Op}(X)$. More concretely, if we understand $\mathrm{Pic}(-)$ on an open covering of X, can we understand it on all of X?

Motivation: In algebraic and complex geometry, the functor $\mathrm{Pic}(-)$ (suitably updated!) is an important invariant of an algebraic variety. If X is a smooth variety, then all elements of $\mathrm{Pic}(X)$ come from subvarieties of codimension 1.

For example if X is a smooth curve, then subvarieties of codimension 1 are points and line bundles on X keep track of zeros and poles of meromorphic functions.

So it would be great if understanding all of this data on an open cover is equivalent to understanding it on the space itself.

Line bundles

What we know: In fact, we do know something about complex line bundles. We know that there is a chain of isomorphisms

$$\mathrm{Pic}(X) \cong [X, B\,\mathrm{GL}_1(\mathbf{C})] = [X, \mathbf{CP}^\infty] \cong H^2_{\mathrm{sing}}(X, \mathbf{Z}).$$

So we observe that

1. Isomorphism classes of line bundles is a homotopy invariant.
2. If the singular cohomology is large then you can expect to have lots of non-trivial line bundles

So armed with this information, we can answer the question of descent asked previously.

Answer: $\mathrm{Pic}(-)$ doesn't satisfy descent in general.

For example, consider $\mathbf{CP}^1 = S^2$ and let 0 and ∞ denote the south and north poles respectively. Let $U_1 = \mathbf{CP}^1 - \{\infty\} \cong \mathbf{C}$ and $U_2 = \mathbf{CP}^1 - \{0\} \cong \mathbf{C}$. Then $U_1 \cap U_2 \cong S^1$. Note that $H^2_{\mathrm{Sing}}(S^1, \mathbf{Z}) = 0$.

So we see that

$$\mathrm{Pic}(U_1) = \mathrm{Pic}(U_2) = \mathrm{Pic}(U_1 \cap U_2) = 0,$$

but

$$\mathrm{Pic}(\mathbf{CP}^1) = H^2(\mathbf{CP}^1, \mathbf{Z}) = \mathbf{Z}[H],$$

where H is a hyperplane section (of $\mathcal{O}(1)$).

Line bundles: complements

A similar argument works in the algebraic situation to prove the failure of descent for the algebraic Picard group. You just have to use that the class group of a unique factorization domain is trivial.

The groupoid of line bundles

Line bundles: Correcting the situation

Upshot: We have discovered that isomorphism classes of line bundles do not satisfy descent in general. What is going wrong?

Line bundles: Correcting the situation

- Line bundles have lots of automorphisms!
- Even though $\mathbf{CP}^1 - \{0\} \times \mathbf{C}$ and $\mathbf{CP}^1 - \{\infty\} \times \mathbf{C}$ restrict to the trivial bundle on $(\mathbf{CP}^1 - \{0\}) \cap (\mathbf{CP}^1 - \{\infty\}) = \mathbf{C}^\times$, the two copies can be glued together using one of these automorphisms.
- Consider gluing $\mathbf{C}^\times \times \mathbf{C}$ along itself via the isomorphism

$$(z, v) \mapsto (z, z^{-1}v).$$

- This gives a different vector bundle on $\mathbf{CP}^1$ than the trivial bundle, namely $\mathcal{O}(-1)$ which is the tautological bundle.

Line bundles: Correcting the situation

- $\mathrm{Pic}(-)$ is not good enough in this situation. It remembers things which are isomorphic, but doesn't really remember why they are isomorphic i.e. it forgets the automorphisms!
- This failure is really telling us that we should study a richer object which remembers all the automorphisms.
- This object is the underlying *groupoid* of $\mathrm{Pic}(-)$.

Line bundles: Correcting the situation

- A *groupoid* is a category in which all morphisms are invertible.
- It is a *categorification* of the idea of taking isomorphism classes of objects.
- Whenever there is a set of isomorphism classes of objects, then there is an underlying groupoid.
- We want to study **Pic** the underlying groupoid of Pic.

The groupoid of line bundles

- We will describe **Pic** as a presheaf of groupoids on $\mathrm{Op}(X)$.
- For each open U, we define $\mathbf{Pic}(U)$ to be the category whose objects are line bundles $\pi \colon \mathcal{L} \to U$ and

$$\mathrm{Hom}(\mathcal{L}, \mathcal{M}) := \left\{ \text{Commuting diagrams } \begin{array}{ccc} \mathcal{L} & \xrightarrow{\sim} & \mathcal{M} \\ \downarrow & & \downarrow \\ U & \xrightarrow{id_U} & U \end{array} \right\}$$

- Note that if $\mathcal{L}, \mathcal{M} \in \mathbf{Pic}(U)$, then the assignment $V \mapsto \mathrm{Hom}(\mathcal{L}|_V, \mathcal{M}|_V)$ is a sheaf on $\mathrm{Op}(U)$.

The groupoid of line bundles

- This is indeed a naive presheaf of groupoids. Namely if $j\colon V \hookrightarrow U$ is a morphism in $\mathrm{Op}(X)$, then

$$j^*\colon \mathbf{Pic}(U) \to \mathbf{Pic}(V)$$

is the induced functor which sends

$$\pi\colon \mathcal{L} \to U \rightsquigarrow \pi|_V\colon \mathcal{L}|_V \to V$$

.

- For a chain of inclusions $W \xrightarrow{i} V \xrightarrow{j} U$ we have

$$i^* \circ j^* = (j \circ i)^*$$

on the nose.

The stack of line bundles

The stack of line bundles

Question:How do we reasonably make this a 'sheaf' of groupoids?

Idea: For an open set $U \in \mathrm{Op}(X)$ and a covering $\bigcup_i U_i$ of U we want to understand vector bundles on U as sections of the groupoid presheaf $\mathbf{Pic}(U_i)$ which agree on $\mathbf{Pic}(U_i \cap U_j)$.

However agreement on $\mathbf{Pic}(U_i \cap U_j)$ is *not* on the nose agreement. In fact we only have agreement upto an isomorphism.

The stack of line bundles

So if $E_i \in \mathbf{Pic}(U_i)$ and $E_j \in \mathbf{Pic}(U_j)$ then agreement in $\mathbf{Pic}(U_i \cap U_j)$ is by definition an isomorphism

$$\sigma_{ij} \colon E_i|_{U_i \cap U_j} \cong E_j|_{U_j \cap U_i}.$$

Is this enough?

No!

The stack of line bundles

If i, j, k we need to make sure that the diagram

$$\begin{tikzcd} E_i|_{U_{ijk}} \arrow[r, "\sigma_{ij}"] \arrow[rr, bend right, "\sigma_{ik}"'] & E_j|_{U_{ijk}} \arrow[r, "\sigma_{jk}"] & E_k|_{U_{ijk}} \end{tikzcd}$$

commutes in $\mathbf{Pic}(U_{ijk})$, where $U_{ijk} = U_i \cap U_j \cap U_k$.

The stack of line bundles

This is enough.

Proposition-Let X be a topological space. Let $\bigcup_{i\in I} U_i = X$ be a cover of X. Suppose we are given vector bundles E_i on U_i

1. isomorphisms
$$\sigma_{ij} \colon E_i|_{U_i\cap U_j} \cong E_j|_{U_j\cap U_i}$$
2. so that on triple intersections
$$E_i|_{U_i\cap U_j\cap U_k} \xrightarrow{\sigma_{ij}} E_j|_{U_i\cap U_j\cap U_k} \xrightarrow{\sigma_{jk}} E_k|_{U_i\cap U_j\cap U_k}$$
so that $\sigma_{jk}\sigma_{ij} = \sigma_{ik}$.

Then there exists a vector bundle E on X so that $E|_{U_i} \cong E_i$.

The proof is left as an exercise.

The stack of line bundles

- So to make this gluing condition precise, we need to organise all these isomorphisms and compatibilities together.
- We can in fact organise all this data in a category called the *category of descent datum*.

The stack of line bundles

Let U be an object of $\mathrm{Op}(X)$ and let $\bigcup_i U_i$ be a cover of U. Then we define a category $\mathbf{Pic}(\{U_i \to U\})$ as follows.

The objects of this category are vector bundles $E_i \in \mathbf{Pic}(U_i)$ and isomorphisms

$$\sigma_{ij} \colon E_i|_{U_{ij}} \cong E_j|_{U_{ij}}$$

satisfying the triple intersection condition. This is the 'descent datum'.

The stack of line bundles

The morphisms are obvious commuting squares i.e. an arrow $\alpha_i \colon E_i \to F_i$ in $\mathbf{Pic}(U_i)$ so that the diagram

$$\begin{tikzcd} E_i \arrow[r, "\sigma_{ij}"] \arrow[d, "\alpha_i"'] & E_j \arrow[d, "\alpha_j"] \\ F_i \arrow[r, "\sigma'_{ij}"] & F_j \end{tikzcd}$$

commutes and also respects triple intersections.

The stack of line bundles

There is a canonical functor

$$\text{can}\colon \mathbf{Pic}(U) \to \mathbf{Pic}(\{U_i \to U\})$$

which sends every vector bundle to it's restriction on the open covering.

This trivially satisfies the 'descent datum' since if E is a vector bundle on U then

- $E|_{U_i \cap U_j} = E|_{U_i \cap U_j}$
- $E|_{U_i \cap U_j \cap U_k} = E|_{U_i \cap U_j \cap U_k} = E|_{U_i \cap U_j \cap U_k}$

Stacks

Stack condition: What we have discussed so far then implies that the functor

$$\mathrm{can}\colon \mathbf{Pic}(U) \to \mathbf{Pic}(\{U_i \to U\})$$

is an equivalence of categories.

Note that we have not discussed whether the functor can is fully faithful, but this follows from the observation that for $\mathcal{L}, \mathcal{M} \in \mathbf{Pic}(U)$ the assignment $V \mapsto \mathrm{Hom}(\mathcal{L}|_V, \mathcal{M}|_V)$ is a sheaf on $\mathrm{Op}(U)$.

Stacks

Stacks

We can now define the stack condition in general

Definition-A stack on $\mathrm{Op}(X)$ is a presheaf of groupoids

$$\mathfrak{X}\colon \mathrm{Op}(X) \to \mathrm{Groupoids}$$

such that for every object $U \in \mathrm{Op}(X)$ and every covering $\{U_i \to U\}_i$ the canonical functor

$$\mathrm{can}\colon \mathfrak{X}(U) \to \mathfrak{X}(\{U_i \to U\}_i)$$

given by restriction is an equivalence of categories.

Stacks

There is a more inspiring way to describe a stack.

Definition- A presheaf of groupoids

$$\mathfrak{X}\colon \ \mathrm{Op}(X) \to \mathrm{Groupoids}$$

is a stack if for all $U \in \mathrm{Op}(X)$ and all coverings $\{U_i \to U\}_i$ the canonical map is an equivalence

$$\mathfrak{X}(U) \to \varprojlim \left(\prod_i \mathfrak{X}(U_i) \rightrightarrows \prod_{ij} \mathfrak{X}(U_{ij}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod_{ijk} \mathfrak{X}(U_{ijk}) \right).$$

This is the correct definition for *higher stacks*, namely for n-stacks you need to consider $n+2$-fold intersections. For ∞-stacks (or, equivalently, derived stacks) you get a cosimplicial object whose limit is what you want (Note that a sheaf would be a 0-stack.)

Complements

Complements

- The failure of $\mathrm{Pic}(-)$ to satisfy descent is closely related to the fact that this functor is *not representable* in the category Top.
- The classifying space $B\,\mathrm{GL}_1(\mathbf{C})$ represents the functor at the level of the homotopy category.
- We have already noted that every representable presheaf is in fact a sheaf.
- This is of course once again related to the existence of automorphisms.

Complements

- Suppose $\mathrm{Pic}(-) \cong \mathrm{Hom}_{\mathsf{Top}}(-, Y)$ for some $Y \in \mathsf{Top}$, then Y carries a universal vector bundle $\pi\colon E_{\mathrm{univ}} \to Y$ corresponding to the element $\mathrm{id}_Y \in \mathrm{Hom}_{\mathsf{Top}}(Y, Y)$ so that for any space $X \in \mathsf{Top}$ and a vector bundle $F \to X$, there exists a morphism $f\colon X \to Y$ so that $f^*E_{\mathrm{univ}} \cong F$.

Complements

- But this cannot be true because then if $i\colon U \to X$ is a trivialising open for F i.e. $F|_U \cong U \times \mathbf{C}$ then the fibre diagram

$$\begin{tikzcd} i^*f^*E_{univ} \arrow[r] \arrow[d] & f^*E_{univ} \arrow[r] \arrow[d] & E_{\mathrm{univ}} \arrow[d] \\ U \arrow[r, "i"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

  implies that $(f \circ i)^* E_{\mathrm{univ}} = U \times \mathbf{C}$ and so $f \circ i\colon U \to Y$ is the constant map. In particular $f\colon X \to Y$ is locally constant, so globally constant since X is connected.

Complements: The algebraic situation

- In the algebraic situation the Zariski topology is too coarse to allow many non-trivial fibrations.
- The Zariski topology for $\mathbf{CP}^1$ is generated by complements of points!
- In this situation one has to work with finer topologies, namely the fpqc and étale topology to answer questions of descent.
- The definition of the topological site easily generalises to the fpqc situation and the étale situation to give the fpqc and étale site respectively.

Complements: Algebraic stacks

- A similar argument as the one given above in algebraic geometry rules out a *fine moduli* for line bundles.
- In the category of schemes over a field k, there exists an algebraic (Artin) stack
$$B\mathbf{G}_m\colon (\mathrm{Sch}_k)_{\acute{e}t} \to \mathrm{Groupoids}$$
which serves as a moduli space for line bundles .
- It was the insight of Deligne, Mumford, Artin and probably others that one can in fact do geometry on this groupoid valued presheaf analogous to the way you do geometry on schemes.

Complements

- A similar trouble with automorphisms also rules out the existence of *fine moduli* for elliptic curves (over schemes).
- Namely, there exist elliptic curves where each fiber is isomorphic, but the elliptic curve is not globally trivial. Such elliptic curves are called *isotrivial*.
- This causes it to fail étale descent, a necessary condition for a functor to be representable.
- To remedy the situation there exists a smooth Deligne-Mumford stack $\mathcal{M}_{1,1}$ over Spec($\mathbf{Z}$) which remembers all these automorphisms (there is also a coarse moduli space, which is modelled by $\mathbf{A}^1_{\mathbf{Z}}$[1]).

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[1] Thanks to Samir Canning for pointing this out

Complements: Advertisements

To learn more about this story of algebraic stacks, JJ and I are planning on organising a learning seminar on algebraic stacks and everyone is welcome to attend and give talks!